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SOLUTION OF NONLINEAR NONCONVEX OPTIMIZATION PROBLEMS BY Ψ–TRANSFORMATION METHOD

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Abstract—The solution of the optimization problems for a wide class of objective functions using the Ψ-Transformation method is considered. The Ψ-Transformation method permits to find the global extremum for a nonconvex differentiable or nondifferentiable function in n-dimensional space. Main principles of the method are presented. Some features of the method connected with the solution of statistical and dynamical optimization problems as well as with the solution of algebraic, transcendental, nonlinear differential and nonlinear partial differential equations are considered. Each problem is illustrated by the corresponding examples.

1. INTRODUCTION

The optimization problems still are of interest. Unfortunately, existing methods allow us to obtain the optimal solution mostly for a narrow class of optimization problems. It is known that the solution of an optimal problem means to find

$$\min F(x) \qquad \text{subject to} \qquad \varphi(x) > c \qquad x \in \mathbf{R}^n \tag{1}$$

or to find

$$\min J[Y(t), U(t), t] \qquad \text{subject to} \qquad \Phi[Y(t), U(t), t] = 0, \tag{2}$$

where $F(x)$ is a function in $\mathbf{R}^n$, x is a vector and $J[Y(t), U(t), t]$ is a functional given in a functional space.

Expression (1) determines a static problem of optimization, expression (2)–a dynamical problem of optimization.

For the solution of problems (1) and (2) by classical methods, it is necessary to satisfy the conditions of differentiability and convexity of the functions $F(x)$, $\varphi(x)$, $J[Y(t), U(t), t]$ and $\Phi[Y(t), U(t), t]$. Unfortunately, these conditions mostly are not satisfied.

If the function $F(x)$ or the functional $J[Y(t), U(t), t]$ is convex and has many extrema it is necessary to find the lowest or the highest value.

Most existing methods of the solution of optimization problems, such as Euler, Lagrange, Kuhn-Tucker methods, are founded on the well known Fermat theorem: at the point of extremum the derivatives are equal to zero. However, if the function is non-differentiable, derivatives at the extremum point cannot be calculated. This leads to additional problems.

In this paper we will present quite different ideas of global solution of optimization problems. These ideas are based on the theory of functions and some properties of Lebesgue measures.

2. PROBLEM FORMULATION

In order to simplify the discussion (without loss of generality), consider the problem of finding

$$\max F(x) \qquad x \in E \tag{3}$$

where E is a closed and bounded subset of $\mathbf{R}^n$. The objective function $F(x)$ may be non-differentiable, nonconvex and may have jump discontinuities (even at the extremum).

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It is assumed that $F(x)$ is bounded and $F(x) \in L_p(E)$. Without loss of generality, we shall also assume that $F(x) \geqslant 0, \quad \forall x \in E$.

We will consider the functions that satisfy the following conditions

1. $F(x)$ achieves its highest (lowest) values only in one point. This means that there is only one global extremum.
2. In the region σ of global extremum

$$\int_\sigma F(x)\,dx > 0 \tag{4}$$

The main idea of the proposed approach is the transformation of the given function $F(x)$ into a set of $n+1$ functions depending on scalar variable ζ.

The transformation is carried out by means of some nonlinear operator:

$$L\{F(x)\} \Rightarrow \Psi(\zeta), \bar{x}_1(\zeta), \dots, \bar{x}_n(\zeta) \tag{5}$$

In order to illustrate this transformation, let us introduce the following notation:

$$E_\zeta^* = \{x : x \in E, \quad F(x) \geqslant \zeta\} \tag{6}$$

where $0 \leqslant \zeta \leqslant \max_{x\in E} F(x)$. If $\zeta = 0$, then, obviously,

$$E_0^* = \{x : x \in E, \quad F(x) \geqslant 0\} = E \tag{7}$$

since $F(x) \geqslant 0$ on E.

For $\zeta > \inf_{x\in E} F(x)$, $E \setminus E_\zeta^* = \bar{E}_\zeta$, where

$$\bar{E}_\zeta = \{x : x \in E, \quad F(x) < \zeta\} \neq \emptyset. \tag{8}$$

Denote by $m(E)$ the measure of E. Then, the measure of E_ζ^* and $\bar{E}_\zeta$ will be $m(E_\zeta^*)$ and $m(\bar{E}_\zeta)$.

Now, let us define the function

$$\Psi(\zeta) = m(E_\zeta^*) \tag{9}$$

It is clear that

$$\Psi(\zeta) = m(E_\zeta^*) = \int \cdots \int_E \theta_\zeta(x_1, \dots, x_n)\,dx_1 \dots dx_n, \tag{10}$$

where $\theta_\zeta(x_1, \dots, x_n)$ is the characteristic function of the set E_ζ^*, that is:

$$\theta_\zeta(x_1, \dots, x_n) = \begin{cases} 1, & \text{if} \quad x \in E_\zeta^* \\ 0, & \text{if} \quad x \in \bar{E}_\zeta \end{cases} \tag{11}$$

It was shown in [1,2,3] that the function $\Psi(\zeta)$ has the following important properties

1. $\Psi(\zeta)$ depends on a scalar variable ζ;
2. $\Psi(\zeta)$ is monotonically decreasing;
3. The minimal value ζ^* for which $\Psi(\zeta^*) = 0$ is equal to the value of the global extremum of $F(x)$.

The function $\Psi(\zeta)$ allows us, in principle, to determine only the value of $F(x)$ at the global extremum. In order to obtain the coordinates of this global extremum, let us define the values

$$\bar{x}_i(\zeta) = \frac{\int \cdots \int_E x_i \theta_\zeta(x_1, \dots, x_n)\,dx_1 \dots dx_n}{\Psi(\zeta)} \tag{12}$$

where $\bar{x}_i(\zeta)$ is the average value of x when $F(x) \geqslant \zeta$. It was proved in [1,4] that:

$$\lim_{\zeta\to\zeta^*} \bar{x}_i(\zeta) = x_i^* \qquad i = 1, \dots, n. \tag{13}$$

As we noted, $\Psi(\zeta^*) = 0$ and x_i^*, $(i = 1, \dots, n)$ are the coordinates of optimal point. Thus, if ζ^* is known, the intersections of vertical line $\zeta = \zeta^*$ with $\bar{x}_i(\zeta)$ on the ζ-plane determine the point x_i^*. It was shown that if $F(x)$ is convex, the convergence of $\bar{x}_i(\zeta)$ to x_i^* is monotonic; if $F(x)$ is not convex, convergence is still attained, but the functions $\bar{x}_i(\zeta)$ may exhibit oscillatory behaviour.

3. ANALYTICAL EXPRESSIONS FOR THE Ψ-TRANSFORMATION

It must be noted that there are very limited possibilities of obtaining an analytical expression for the Ψ-Transformation. This can be accomplished only in very simple cases.

As it was noted before, the Ψ-Transformation is based on the relationships between the measure of the set E_ζ^* and the value of ζ. In principle, it is possible to determine the zeros of the function $F(x) - \zeta$, which define the boundary of the set E_ζ^* and then to evaluate the measure of this set.

If ζ varies to $\zeta + \Delta\zeta$, then in order to define $\Psi(\zeta + \Delta\zeta)$, we must construct the new set $E_{\zeta+\Delta\zeta}^*$ and perform a new integration. Thus, the application of the operator L to $F(x)$ would require multi-dimensional integrations over a region which varies with ζ.

This leads to difficult analytical problems, but sometimes it is possible to derive the analytical expression of $\Psi(\zeta)$. The following examples show these possibilities.

Let

$$F(x) = -a_1x^2 + a_2x + a_3 \tag{14}$$

In order to obtain the analytical expression of $\Psi(\zeta)$, we have to evaluate, for any given value of $\zeta > 0$, the roots x_1, x_2 of the equation

$$-a_1x^2 + a_2x + a_3 - \zeta = 0 \tag{15}$$

Then, the measure of the set E_ζ^* is given by $|x_1 - x_2|$ and

$$\Psi(\zeta) = \sqrt{\alpha - \beta\zeta}, \tag{16}$$

where

$$\alpha = \frac{a_2^2 + 4a_1a_3}{a_1^2}, \qquad \beta = \frac{4}{a_1}. \tag{17}$$

As we know, the optimum value of $F(x)$ is equal to the value ζ^* for which $\Psi(\zeta^*) = 0$. In fact

$$\zeta^* = \max F = \frac{\alpha}{\beta} = \frac{a_2^2}{4a_1} + a_3. \tag{18}$$

The same results will be obtained if we use the conventional method

$$\frac{dF}{dx} = -2a_1x + a_2 = 0, \qquad x = \frac{a_2}{2a_1}$$

and

$$\max F = \frac{a_2^2}{4a_1} + a_3.$$

Let us consider now an example when a function $F(x)$ is given on the plane $0x_1x_2$:

$$F(x_1, x_2) = c^2[1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}] \tag{19}$$

We know that ζ is the scalar value of F and $F(x_1, x_2) = \zeta$. Consequently

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{\zeta}{c^2} = 1, \tag{20}$$

which is equivalent to

$$\frac{x_1^2}{A^2} + \frac{x_2^2}{B^2} = 1, \tag{21}$$

where

$$A = \frac{a}{c}\sqrt{c^2 - \zeta}, \qquad B = \frac{b}{c}\sqrt{c^2 - \zeta}. \tag{22}$$

In this case, $m(E_\zeta^*)$ is the surface area of the region whose boundary is defined by the intersection of $F(x_1, x_2)$ with a plane parallel to the plane $0x_1x_2$ and located at distance ζ. It is known that

$$m(E_\zeta^*) = \oint x_1\,dx_2 - \oint x_2\,dx_1$$

Now, denoting

$$x_1 = A\cos t, \qquad x_2 = B\sin t,$$

we have

$$\Psi(\zeta) = m(E_\zeta^*) = \frac{1}{2}\int_0^{2\pi} (AB\cos^2 t + AB\sin^2 t)\,dt = \frac{1}{2}\int_0^{2\pi} AB\,dt = \pi AB$$

or equivalently,

$$m(E_\zeta^*) = \Psi(\zeta) = \pi ab\frac{c^2-\zeta}{c^2} \tag{23}$$

By imposing $\Psi(\zeta^*) = 0$, we obtain

$$\max F(x_1, x_2) = \zeta^* = c^2. \tag{24}$$

4. CONSTRUCTION OF THE TRANSFORMED FUNCTIONS

As was noted before, there are considerable difficulties, in general, to obtain the analytical expressions of the functions $\Psi(\zeta)$, $\bar{x}_i(\zeta)$ $(i = 1, \dots, n)$. Nevertheless, these functions can be determined point by point and by approximation of these points we can define the functions $\Psi(\zeta)$ and $\bar{x}_i(\zeta)$ $(i = 1, \dots, n)$. Let $m(E) = 1$. The function $\Psi(\zeta)$ can be expressed by (10) where the integral may be calculated by the Monte-Carlo method. It is more convenient to define $\Psi(\zeta)$ by the equation

$$\Psi(\zeta) = \int\limits_E \cdots \int [F(x_1, \dots, x_n) - \zeta]^\ell \theta_\zeta(x_1, \dots, x_n)\,dx_1 \dots dx_n, \tag{25}$$

and

$$\bar{x}_i(\zeta) = \frac{\int\limits_E \cdots \int [F(x_1, \dots, x_n) - \zeta]^\ell \theta_\zeta(x_1, \dots, x_n)\,dx_1 \dots dx_n}{\Psi(\zeta)} \tag{26}$$

because in this case the functions $\Psi(\zeta)$ and $\bar{x}_i(\zeta)$ are more smooth [1].

The functions $\Psi(\zeta)$ and $\bar{x}_i(\zeta)$ may be defined by the simpler expressions

$$\Psi_\nu(\zeta) = \frac{\xi_\nu}{s}, \tag{27}$$

$$\bar{x}_i^\nu = \frac{\sum_{j=1}^{\xi_\nu} x_i^j}{\xi_\nu}, \tag{28}$$

where ξ_ν is the number of events in statistical tests, where $F(x_1, \dots, x_n) \geqslant \zeta_\nu$, and s is the number of statistical tests. We use these statistical tests for implementation of the Monte-Carlo method.

In practice, it is impossible to determine directly, with sufficient accuracy, the values ζ^* for which $\Psi(\zeta^*) = 0$ since in the neighborhood of this point, the probability of realizing the event $F(x) \geqslant \zeta$ is near zero. The way out is to approximate the functions $\Psi(\zeta), \bar{x}_i(\zeta)$ $(i = 1, \dots, n)$ with curves determined from points calculated for $\zeta < \zeta^*$. Subsequently, these curves are extrapolated and the values ζ^*, $\bar{x}_i(\zeta^*)$ $(i = 1, \dots, n)$ are evaluated with more accuracy.

The extrapolation is obtained by polynomials. For the function $\Psi(\zeta)$ in most cases we used the parabolic approximation. The coefficients of polynomials are defined by the well known least-squares method.

Now we present the algorithm

1) By random search with equal probability choose the values of $x_1, \ldots, x_n$.
2) Calculate the value of the function $F(x_1, \ldots, x_n)$.
3) Repeat steps 1) and 2) s-times.
4) The maximum $\sup F$ and minimum $\inf F$ should be determined from s values of $F(x_1, \ldots, x_n)$.
5) The interval $[(\sup F - \inf F)/2, \sup F]$ is divided into k equal subintervals.
6) The values ζ_ν, $(\nu = 1, 2, \ldots, k)$ and Ψ_ν are defined by the formulae

$$\zeta_\nu = \frac{\sup F - \inf F}{2} + (\nu - 1)\Delta\zeta \tag{29}$$

$$\Psi_\nu = \frac{\xi_\nu}{s}, \qquad \Delta\zeta = \zeta_\nu - \zeta_{\nu-1}, \tag{30}$$

where ξ_ν is the number of events when inequalities $F(x_1, \ldots, x_n) \geqslant \zeta_\nu$ are satisfied.

7) According to

$$M_1[\zeta] = \frac{1}{k}\sum_{\nu=1}^{k}\zeta_\nu, \qquad M_1[\Psi] = \frac{1}{k}\sum_{\nu=1}^{k}\Psi(\zeta_\nu),$$
$$M_2[\zeta] = \frac{1}{k}\sum_{\nu=1}^{k}\zeta_\nu^2, \qquad M_{1,1}[\zeta,\Psi] = \frac{1}{k}\sum_{\nu=1}^{k}\zeta_\nu\Psi(\zeta_\nu),$$
$$M_3[\zeta] = \frac{1}{k}\sum_{\nu=1}^{k}\zeta_\nu^3, \qquad M_{2,1}[\zeta,\Psi] = \frac{1}{k}\sum_{\nu=1}^{k}\zeta_\nu^2\Psi(\zeta_\nu),$$
$$M_4[\zeta] = \frac{1}{k}\sum_{\nu=1}^{k}\zeta_\nu^4,$$

the parametres of the linear system are determined:

$$\begin{aligned} M_4[\zeta]\alpha_0 + M_3[\zeta]\alpha_1 + M_2[\zeta]\alpha_2 &= M_{2,1}[\zeta,\Psi] \\ M_3[\zeta]\alpha_0 + M_2[\zeta]\alpha_1 + M_1[\zeta]\alpha_2 &= M_{1,1}[\zeta,\Psi] \\ M_2[\zeta]\alpha_0 + M_1[\zeta]\alpha_1 + M_0[\zeta]\alpha_2 &= M_{0,1}[\zeta,\Psi]. \end{aligned} \tag{31}$$

One should note that

$$M_0[\zeta] = 1, \qquad M_{0,1}[\zeta,\Psi] = M_1[\Psi]. \tag{32}$$

8) By solving the last system of linear equations, the coefficients $\alpha_0, \alpha_1, \alpha_2$ of the parabola

$$\varphi(\zeta) = \alpha_0\zeta^2 + \alpha_1\zeta + \alpha_2 \tag{33}$$

are defined.

9) The roots of the least square equation are calculated.
10) The minimal of these roots will be equal to the scalar value ζ^* of the global extremum of $F(x_1, \ldots, x_n)$.
11) The average value of x_i when $F(x_1, \ldots, x_n) \geqslant \zeta_\nu$ for each ζ_ν, $(\nu = 1, \ldots, k)$ are defined by the formula

$$\bar{x}_i^\nu = \frac{1}{\xi_\nu}\sum_{j=1}^{\xi_\nu} x_i^j. \tag{34}$$

12) According to

$$M_{1,1}[\zeta,\bar{x}_i(\zeta)] = \frac{1}{k}\sum_{\nu=1}^{k}\zeta_\nu\bar{x}_i(\zeta_\nu)$$

$$M_{2,1}[\zeta,\bar{x}_i(\zeta)] = \frac{1}{k}\sum_{\nu=1}^{k}\zeta_\nu^2\bar{x}_i(\zeta_\nu)$$

$$M_{0,1}[\zeta,\bar{x}_i(\zeta)] = \frac{1}{k}\sum_{\nu=1}^{k}\bar{x}_i(\zeta_\nu) = M_1[\bar{x}_i(\zeta)]$$

the system of equations below should be solved

$$\begin{aligned} M_4[\zeta]\beta_0^i + M_3[\zeta]\beta_1^i + M_2[\zeta]\beta_2^i &= M_{2,1}[\zeta,\bar{x}_i(\zeta)] \\ M_3[\zeta]\beta_0^i + M_2[\zeta]\beta_1^i + M_1[\zeta]\beta_2^i &= M_{1,1}[\zeta,\bar{x}_i(\zeta)] \\ M_2[\zeta]\beta_0^i + M_1[\zeta]\beta_1^i + M_0[\zeta]\beta_2^i &= M_{0,1}[\zeta,\bar{x}_i(\zeta)] \end{aligned} \tag{35}$$

The values $M_4[\zeta]$, $M_3[\zeta]$ and $M_2[\zeta]$ are defined from the step 7).

13) The equation

$$x_i^*(\zeta) = \beta_0^i\zeta^{*2} + \beta_1^i\zeta^* + \beta_2^i \tag{36}$$

is then solved.

14) Each value x_i^*, $(i = 1,\dots,n)$ must be put into objective function

$$F(x_1^*,\dots,x_n^*) = F^* \tag{37}$$

15) If $F^* = \zeta^*$, the solution is correct. If $|F^* - \zeta^*| > \epsilon$ the process must be repeated.

For $\Psi(\zeta)$ and $\bar{x}_i(\zeta)$ it is sometimes better to use the expressions (25),(26).

In this case only the steps 6) and 11) of the algorithm should be changed. The new steps are as follows

6*) The values ζ_ν, $(\nu = 1,2,\dots,k)$ and Ψ_ν are defined by the formulas

$$\zeta_\nu = \frac{\sup F - \inf F}{2} + (\nu-1)\Delta\zeta, \qquad \Psi_\nu = \frac{\sum_{j=1}^{\xi_\nu}|F(x_1,\dots,x_n) - \zeta_\nu|^\ell}{s} \qquad \ell = 1,2,\dots \tag{38}$$

11*) The mean values of x_i when $F(x_1,\dots,x_n) \geqslant \zeta_\nu$ are defined by the formulas

$$\bar{x}_i = \frac{\sum_{j=1}^{s} x_i^j|F(x^j) - \zeta|^\ell\theta(x^j,\zeta)}{\sum_{j=1}^{s}|F(x^j) - \zeta|^\ell\theta(x^j,\zeta)}, \qquad i = 1,\dots,n \tag{39}$$

There are some possibilities of reduction of errors.

The so-called method of variations in the Ψ-plane is used for reducing the errors due to the approximation procedure. The approach is based on the consideration of local variations in the neighborhoods of the optimal values x_i^* $(i = 1,\dots,n)$, ζ^*, until the condition $F(x_1^*,\dots,x_n^*) = \zeta^*$ is satisfied with the required accuracy. It was shown that a sequential process can be performed, which converges to the global extremum [1].

Now we illustrate by examples the solution of some optimal problems.

5. NUMERICAL SOLUTION OF VARIOUS PROBLEMS BY THE Ψ-TRANSFORMATION

5.1 Nonlinear Programming Problem

We consider in [1] the minimization of the cost of a launch of a space vehicle [9].

The variables of the objective function are physical parameters of the launch vehicle such as the weights of stages of a rocket, mass fraction, thrusts of stages, etc. Objective function is nonlinear and contains 25 parameters. There are 25 nonlinear constraints.

The problem was solved by the Ψ-Transformation, and the minimal value of objective function was equal to

$$F^*(x_1,\dots,x_{25}) = 1421.9 \text{ million dollars.}$$

For details see [1], pp. 133–141.

5.2 The Dynamic Problems of Optimization

Let us consider the problem of automatic landing of an aircraft.

The problem is to find such control function $u(t)$ that minimizes the difference between the ideal trajectory and the realized h_0 trajectory of the landing plane. The ideal trajectory $h(t)$ of landing is, [1]:

$$h(t) = \begin{cases} 100e^{-t/5}, & \text{when } 0 \leqslant t < 15 \\ 20 - t, & \text{when } 15 \leqslant t \leqslant 20 \end{cases} \tag{40}$$

We have to find the control function $u(t)$ that minimizes the error

$$e(t) = [h(t) - h_0(t)]^2 + [\dot{h}(t) - \dot{h}_0(t)]^2. \tag{41}$$

In this case we have to minimize the expression

$$I = \int_0^{t_f} e(t)\, dt, \tag{42}$$

or

$$I = \int_0^{t_f} \left\{ [h(t) - h_0(t)]^2 + [\dot{h}(t) - \dot{h}_0(t)]^2 \right\} dt \tag{43}$$

subject to

$$\begin{aligned} \ddot{\theta}(t) &= -0.6\,\dot{\theta}(t) - 0.72\,\theta(t) + 0.00296875\,h(t) - 2.375\,u(t) \\ \dot{h}(t) &= 102.4\,\theta(t) - 0.4\,h(t) \end{aligned}$$

$$\dot{h}(t) < 0, \qquad h(t) > 0, \qquad -35^\circ \leqslant u \leqslant 15^\circ, \qquad 0^\circ \leqslant u(t_f) \leqslant 10^\circ$$

where h is the altitude of the plane and θ is the angle. This problem was solved by Ψ-Transformation method. The $e(t)$ on the interval $[0, t_f]$ was less than 1%.

5.3 The Solution of Algebraic, Transcendental and Nonlinear Differential Equations

It is known that if there is a system of nonlinear or transcendental equations, the solution of this system is possible by minimizing a certain function $\Phi(x)$. Suppose that there is a system of nonlinear or transcendental equations

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ \vdots \qquad & \quad \vdots \\ f_n(x_1, \ldots, x_n) &= 0 \end{aligned} \tag{44}$$

We consider the function

$$\Phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} [f_i(x_1, \ldots, x_n)]^2 \tag{45}$$

The coordinates $(x_1, \ldots, x_n)$ of global extremum of $\Phi(x_1, \ldots, x_n)$ are the roots of the given system of nonlinear or transcendental equations. Suppose we have the system of transcendental equatons

$$\begin{aligned} x_1x_3 + (x_3 + x_5^2)x_4 + (x_6 + x_7)^2 &= 20 \\ x_1^2 + (2x_4 + x_6^2) - x_3 - (x_7^2 + x_8^2) &= 178 \\ x_1x_3 + 7x_5 - (x_4 + 2x_6 + x_2^2) &= -25 \\ x_1x_4 - x_3^3 + (x_2 - x_1 - x_8^2) + 20\cos(x_6) &= -5.07287 \\ x_1x_2 - 10x_4^2 - (x_5^2 - x_7)^2 + 30\sin(x_3) &= 22.76639 \\ x_1^3x_5x_3 - x_6x_7 + x_4x_8 - x_2^2 &= -126 \\ x_1^2 - (x_3 + x_5 + x_7)^2 + x_8^2 + x_6^2 + \cos(x_4) &= 45.5403 \\ 4x_1 - 8x_3^2 + (x_4^2 + x_2 - x_6)(3x_7 - x_8^2) &= -364 \end{aligned}$$

This system was solved by Ψ-Transformation method. The solution is

$$x^* = (1.998, 7.005, -3.008, -0.974, 2.999, -3.993, 5.009) \qquad \text{Maximum error is } 0.001$$

Let us consider now the solution of a nonlinear differential equation. The solution of such an equation we will seek in the form of some polynomial series. The parameters of this series will be found by the Ψ-Transformation method. We define such function $y(t)$, for which the maximal deviation from the correct solution would be minimal. This minimization is carried out by the Ψ-Transformation method.

Suppose we have the nonlinear differential equation

$$(\sin(t) + y)\dot{y} - \cos(t)y^2 - 3\,t^5 y - \tan(t) = 0 \tag{46}$$

$$y(0) = 0.484 \tag{47}$$

The solution obtained by the Ψ-Transformation method is

$$y(t) = 0.484 + 0.484\,t + 0.11\,t^{1.2} - 1.976\,t^{1.6} + 2.249\,t^{2.001} + 0.324\,t^5 + 0.0075\,t^9$$

The maximum deviation is $\Delta_{max} = 0.007$.

Next example:

$$y(\ddot{y})^2 y^{(5)} - 45\,\ddot{y}\,\dddot{y} + 40(\dddot{y})^3 = 0 \tag{48}$$

$$y(0) = 0, \quad \dot{y}(0) = 1, \quad \ddot{y}(0) = 1, \quad \dddot{y}(0) = 0, \quad y^{(4)}(0) = -1 \tag{49}$$

The solution by the Ψ-Transformation:

$$y(t) = -t + 0.5t^2 - 0.0416667t^4 - 0.003166t^{4.293} + 0.0041771t^{4.535} - 0.022841t^{5.697} + 0.028857t^{5.957} - 0.009588\,t^{6.669} + 0.001692\,t^{7.519} + 0.0011534\,t^{8.208} - 0.000864\,t^{8.568} - 0.0000722\,t^{8.923}$$

The maximum deviation is $\Delta_{max} = 0.005$.

Finally we consider the solution of the partial differential equation:

$$\begin{aligned} 2u\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \\ u(0,t) &= 0.5\,t^2 \\ u(x,0) &= 2x \end{aligned} \tag{50}$$

The solution by the Ψ-Transformation is

$$u(x,t) = cx + bt^2 + \sum_{i=1}^{10} \alpha_i x^{\beta_i}(1-x)^{\gamma_i} t^{\xi_i}, \qquad c = 2,\ b = 0.5,$$

where $\alpha_i, \beta_i, \gamma_i, \xi_i$, are given in Table 1.

Table 1

i	α_i	β_i	γ_i	ξ_i
1	-0.23922	2	3	4
2	0.2340816	3	4	5
3	0.4532139	5	5	6
4	0.2340816	6	6	7
5	-0.7879126	7	7	8
6	-0.2668861	8	8	9
7	-0.134642	10	9	10
8	-0.3582041	12	10	12
9	-0.3048604	13	11	12
10	0.2388556	14	12	13

The solution of the partial differential equation was carried out by G. Chumburidze.

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